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Abstract

The purpose of this note is to establish a probabilistic representation formula for Navier–Stokes equations on a compact Riemannian manifold. To this end, we first give a geometric interpretation of Constantin and Iyer's representation formula for the Navier–Stokes equation, then extend it to a compact Riemannian manifold. We shall use Elworthy–Le Jan–Li's idea to decompose de Rham–Hodge Laplacian operator on a manifold as a sum of the square of vector fields.

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1 Introduction

The Navier–Stokes equations on $\mathbb{R}^n$ or on a torus $\mathbb{T}^n$,

$$\begin{cases} \partial_t u + (u\cdot\nabla)u - \nu\Delta u + \nabla p = 0, \\ \nabla\cdot u = 0, \quad u|_{t=0} = u_0, \end{cases} \tag{1.1}$$

describe the evolution of the velocity u of an incompressible viscous fluid with kinematic viscosity $\nu > 0$, as well as the pressure p. Such equations attract always the attention of many researchers, with an enormous quantity of publications in the literature. Concerning classical results about (1.1), we refer to the book [24]. The Lagrangian description of the fluid is to determine the position at time t of the particle of fluid. Due to its high nonlinearity, such a description was not used too often in the past. However, since the seminal works [12] on the resolution of ordinary differential equations with coefficients of low regularity and [6] on the relaxed variational principle for Euler equations, there are more and more interests in Lagrangian descriptions. We refer to [1, 14, 15, 26, 27] for new developments and various generalizations of [12], to [7, 2] for generalized flows to Euler equations and to [3, 4, 5] for generalized flows to Navier–Stokes equations.

Connections between Navier–Stokes equations and stochastic evolution have a quite long history: it can be traced back to a work of Chorin [9]. In [20], Le Jan and Sznitman used a backward-in-time branching process to express Navier–Stokes equations through Fourier

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transformations. In [8], a representation formula using noisy flow paths for 3-dimensional Navier–Stokes equation was obtained. An achievement has been realized by Constantin and Iyer in [11] by using stochastic flows. We also refer to [11] for a more complete description on the history of the developments.

For reader's convenience, let us first state Constantin and Iyer's result [11]:

Theorem 1.1 (Constantin–Iyer). *Let $\nu > 0$, W be an n-dimensional Wiener process, $k \geq 1$, and $u_0 \in C^{k+1,\alpha}$ a given deterministic divergence-free vector field. Let the pair (X, u) satisfy the stochastic system*

$$\begin{cases} \mathrm{d}X_t = \sqrt{2\nu}\,\mathrm{d}W_t + u_t(X_t)\,\mathrm{d}t, \\ u_t = \mathbb{E}\mathbf{P}\big[(\nabla X_t^{-1})^*(u_0 \circ X_t^{-1})\big], \end{cases} \tag{1.2}$$

where $\mathbf{P}$ is the Leray–Hodge projection and the star $*$ denotes the transposed matrix. Then u satisfies the incompressible Navier–Stokes equations (1.1)*.*

Using this stochastic representation, Constantin and Iyer were able to give a self-contained proof of the local existence of the solution to the system (1.1). Two proofs of Theorem 1.1 were provided in [11]: the first one uses heavily the fact that the diffusion coefficient of the stochastic differential equation (SDE) in (1.2) is constant, and transforms it into a random ODE by absorbing the Wiener process into the drift coefficient u; the second one applies the generalized Itô formula to the quantity $\big(\nabla X_t^{-1}\big)^*\big(u_0 \circ X_t^{-1}\big)$ which, combined with the stochastic PDE fulfilled by the inverse X_t^{-1}, leads to the desired result. Note that if $x \to u_t(x)$ is 2π-periodic with respect to each component, then SDE (1.2) defines a flow of diffeomorphims of the torus $\mathbb{T}^n$. For the sake of simplicity, we only consider this last situation in Section 2.

In order to avoid the computation of the inverse X_t^{-1} of X_t, X. Zhang used in [25] the idea that the inverse flow can be described by SDEs driven by time-reversed Brownian motion; he established a similar representation formula for the backward incompressible Navier–Stokes equations.

In this note, we first give in Section 2 a more geometric interpretation to the formula of u_t in Theorem 1.1, then provide an alternative proof using directly Kunita's formula ([17, p.265, Theorem 2.1]) for the pull-back of vector fields under the stochastic flow: surprisingly enough, it is simpler to use the inverse flow. More precisely, we get the following expression

$$\int_{\mathbb{T}^n} \langle u_t, v\rangle\,\mathrm{d}x = \mathbb{E}\bigg(\int_{\mathbb{T}^n} \langle u_0, (X_t^{-1})_* v\rangle\,\mathrm{d}x\bigg), \quad \forall\, t \geq 0, \tag{1.3}$$

which means that the evolution of u_t in the direction v is equal to the average of the evolution of v under the inverse flow X_t^{-1} in the initial direction u_0. The purpose of Section 3 is to establish a stochastic representation formula for the Navier–Stokes equations on a compact Riemannian manifold M, where the difficulty is to deal with the de Rham–Hodge Laplacian operator $\square$. We shall use the idea in [13] to decompose $\square$ as a sum of the square of Lie derivatives: $\square = \sum_{i\in\mathcal{I}} \mathcal{L}_{A_i}^2$, where the family $\mathcal{I}$ could be finite or countable. In general, the vector fields A_i are not of divergence free. See Section 3 for the conditions on $\{A_i;\, i \in \mathcal{I}\}$ which ensure such a decomposition. A new formula in Section 3 is

$$u_t = \mathbb{E}\Big[\mathbf{P}\big(\rho_t\,(X_t^{-1})^* u_0^*\big)^{\#}\Big] \tag{1.4}$$

where ρ_t is the density of the associated stochastic flow X_t, and we use $*$ to transform a vector field to a differential form, $\#$ to transform a differential form to a vector field.

In Section 4, we shall treat two important examples: tori and spheres for which we prove that the divergence-free eigenvector fields of $\square$ enjoy all required properties in Section

3. Therefore, they will generate volume-preserving stochastic flows for which Formula (1.4) holds with $\rho_t = 1$. Finally, in Section 5 we shall present some explicit computations in the case of the sphere, to exhibit the properties of $\{A_i;\, i \in \mathcal{I}\}$ used in Section 3.

2 An alternative proof of Constantin–Iyer's result

Before giving the proof, let us prepare some materials. Let M be a compact Riemannian manifold without boundary and $\varphi : M \to M$ a diffeomorphism. Given a vector field A on M, the pull-back vector field $\varphi_*^{-1}(A)$ is defined by

$$\big(\varphi_*^{-1}(A)f\big)(x) = A(f \circ \varphi^{-1})(\varphi(x)), \quad \text{for any} \quad f \in C^1(M), x \in M.$$

Equivalently,

$$\varphi_*^{-1}(A)(x) = d\varphi^{-1}(\varphi(x))A(\varphi(x)) = (d\varphi(x))^{-1}A(\varphi(x)), \tag{2.1}$$

where $d\varphi$ is the differential of φ. For two smooth vector fields A, B on M, the Lie derivative $\mathcal{L}_A B$ is defined as

$$(\mathcal{L}_A B)(x) = \lim_{t\to 0} \frac{\varphi_{t*}^{-1}(B)(x) - B(x)}{t},$$

where φ_t is the flow generated by A and $\varphi_{t*}^{-1}(B) = (\varphi_t)_*^{-1}(B)$. It is well known that $\mathcal{L}_A B = [A, B] = AB - BA$. We have the following simple result.

Lemma 2.1. *If A and B are vector fields of divergence free on M, then so is $\mathcal{L}_A B$.*

Proof. We can provide two different proofs. (i) Since the vector fields A and B are of divergence free, it holds that $\int_M Af \,\mathrm{d}x = \int_M Bf \,\mathrm{d}x = 0$ for any function $f \in C^1(M)$. Therefore,

$$\int_M (\mathcal{L}_A B)f \,\mathrm{d}x = \int_M A(Bf)\,\mathrm{d}x - \int_M B(Af)\,\mathrm{d}x = 0.$$

This clearly implies that $\mathcal{L}_A B$ is of divergence free.

(ii) By the definition of $\mathcal{L}_A B$, it suffices to show that $\varphi_{t*}^{-1}(B)$ is of divergence free for all $t \geq 0$. To this end, take any $f \in C^1(M)$, we have

$$\int_M \big\langle \nabla f, \varphi_{t*}^{-1}(B)\big\rangle \,\mathrm{d}x = \int_M \big(\varphi_{t*}^{-1}(B)f\big)(x)\,\mathrm{d}x = \int_M B(f \circ \varphi_t^{-1})(\varphi_t(x))\,\mathrm{d}x.$$

Since A is of divergence free, the flow φ_t preserves the volume measure of M. Thus

$$\int_M \big\langle \nabla f, \varphi_{t*}^{-1}(B)\big\rangle \,\mathrm{d}x = \int_M B(f \circ \varphi_t^{-1})(y)\,\mathrm{d}y = 0$$

since the vector field B is also of divergence free. The above equality implies that $\varphi_{t*}^{-1}(B)$ is of divergence free. □

Now we present another proof of Theorem 1.1, using directly [17, Theorem 2.1, p.265]. Note that for the part we use in this theorem, it is sufficient that u_t is of $C^{2,\alpha}$ which insures that X_t is a flow of diffeomorphisms.

Proof of Theorem 1.1. Let (X,u) be the pair solving the system (1.2). Then $X=(X_t)_{t\geq 0}$ is a stochastic flow of C^2-diffeomorphisms on $\mathbb{T}^n$. Since the diffusion coefficient of the SDE is constant and the drift u is of divergence free, we know that the flow X_t preserves the volume measure of the torus $\mathbb{T}^n$. Let v be a vector field of divergence free on $\mathbb{T}^n$, we have by the expression of u in (1.2) that

$$\begin{aligned}\int_{\mathbb{T}^n}\langle u_t,v\rangle\,\mathrm{d}x&=\mathbb{E}\bigg(\int_{\mathbb{T}^n}\big\langle(\nabla X_t^{-1})^*(u_0\circ X_t^{-1}),v\big\rangle\,\mathrm{d}x\bigg)\\&=\mathbb{E}\bigg(\int_{\mathbb{T}^n}\big\langle u_0\circ X_t^{-1},(\nabla X_t^{-1})v\big\rangle\,\mathrm{d}x\bigg)\\&=\mathbb{E}\bigg(\int_{\mathbb{T}^n}\big\langle u_0,(\nabla X_t^{-1}(X_t))v(X_t)\big\rangle\,\mathrm{d}x\bigg),\end{aligned}$$

where in the last equality we have used the measure-preserving property of X_t^{-1}. According to (2.1), we get

$$\int_{\mathbb{T}^n}\langle u_t,v\rangle\,\mathrm{d}x=\mathbb{E}\bigg(\int_{\mathbb{T}^n}\big\langle u_0,X_{t*}^{-1}(v)\big\rangle\,\mathrm{d}x\bigg),\quad\forall\, t\geq 0.\tag{2.2}$$

The formula (2.2) means that the evolution of u_t in the direction v is equal to the average of the evolution of v under the inverse flow X_t^{-1} in the initial direction u_0.

Now by [17, p.265], if u_t is of $C^{1,\alpha}$, we have

$$X_{t*}^{-1}(v)=v+\sqrt{2\nu}\sum_{i=1}^n\int_0^t X_{s*}^{-1}(\partial_i v)\,\mathrm{d}W_s^i+\nu\int_0^t X_{s*}^{-1}(\Delta v)\,\mathrm{d}s+\int_0^t X_{s*}^{-1}([u_s,v])\,\mathrm{d}s,$$

where $\partial_i v$ denotes the partial derivative of v. Substituting this expression of $X_{t*}^{-1}(v)$ into (2.2), we arrive at

$$\begin{aligned}\int_{\mathbb{T}^n}\langle u_t,v\rangle\,\mathrm{d}x=\int_{\mathbb{T}^n}\langle u_0,v\rangle\,\mathrm{d}x&+\nu\mathbb{E}\int_0^t\!\!\int_{\mathbb{T}^n}\big\langle u_0,X_{s*}^{-1}(\Delta v)\big\rangle\,\mathrm{d}x\mathrm{d}s\\&+\mathbb{E}\int_0^t\!\!\int_{\mathbb{T}^n}\big\langle u_0,X_{s*}^{-1}([u_s,v])\big\rangle\,\mathrm{d}x\mathrm{d}s.\end{aligned}\tag{2.3}$$

As the vector field Δv is of divergence free, we have by (2.2) that

$$\mathbb{E}\int_0^t\!\!\int_{\mathbb{T}^n}\big\langle u_0,X_{s*}^{-1}(\Delta v)\big\rangle\,\mathrm{d}x\mathrm{d}s=\int_0^t\!\!\int_{\mathbb{T}^n}\langle u_s,\Delta v\rangle\,\mathrm{d}x\mathrm{d}s.\tag{2.4}$$

Next by Lemma 2.1, we know that $[u_s,v]$ is also of divergence free, therefore again by (2.2),

$$\begin{aligned}\mathbb{E}\int_0^t\!\!\int_{\mathbb{T}^n}\big\langle u_0,X_{s*}^{-1}([u_s,v])\big\rangle\,\mathrm{d}x\mathrm{d}s&=\int_0^t\!\!\int_{\mathbb{T}^n}\langle u_s,[u_s,v]\rangle\,\mathrm{d}x\mathrm{d}s\\&=\int_0^t\!\!\int_{\mathbb{T}^n}\langle u_s,(u_s\cdot\nabla)v-(v\cdot\nabla)u_s\rangle\,\mathrm{d}x\mathrm{d}s\\&=\int_0^t\!\!\int_{\mathbb{T}^n}\langle u_s,(u_s\cdot\nabla)v\rangle\,\mathrm{d}x\mathrm{d}s-\frac{1}{2}\int_0^t\!\!\int_{\mathbb{T}^n}(v\cdot\nabla)|u_s|^2\,\mathrm{d}x\mathrm{d}s\\&=\int_0^t\!\!\int_{\mathbb{T}^n}\langle u_s,(u_s\cdot\nabla)v\rangle\,\mathrm{d}x\mathrm{d}s,\end{aligned}$$

where in the last equality we have used the fact that v is of divergence free. Substituting this equality and (2.4) into (2.3), we obtain for all $t\geq 0$ that

$$\int_{\mathbb{T}^n}\langle u_t,v\rangle\,\mathrm{d}x=\int_{\mathbb{T}^n}\langle u_0,v\rangle\,\mathrm{d}x+\nu\int_0^t\!\!\int_{\mathbb{T}^n}\langle u_s,\Delta v\rangle\,\mathrm{d}x\mathrm{d}s+\int_0^t\!\!\int_{\mathbb{T}^n}\langle u_s,(u_s\cdot\nabla)v\rangle\,\mathrm{d}x\mathrm{d}s.$$

The above equality implies that for a.e. $t \geq 0$, it holds

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{T}^n}\langle u_t, v\rangle\,\mathrm{d}x = \nu\int_{\mathbb{T}^n}\langle u_t, \Delta v\rangle\,\mathrm{d}x + \int_{\mathbb{T}^n}\langle u_t, (u_t\cdot\nabla)v\rangle\,\mathrm{d}x.$$

Multiplying both sides by a real-valued function $\alpha \in C_c^1([0,\infty))$ and integrating by parts, we arrive at

$$\alpha(0)\int_{\mathbb{T}^n}\langle u_0, v\rangle\,\mathrm{d}x + \int_0^\infty\int_{\mathbb{T}^n}\big[\alpha'(t)\langle u_t, v\rangle + \nu\alpha(t)\langle u_t, \Delta v\rangle + \alpha(t)\langle u_t, (u_t\cdot\nabla)v\rangle\big]\,\mathrm{d}x\mathrm{d}t = 0.$$

This implies that u_t solves strongly the Navier–Stokes equation, since u_t was assumed to be of $C^{2,\alpha}$. $\square$

3 Extension to compact Riemannian manifolds

In this section, we shall establish the stochastic representation for Navier–Stokes equations on a compact Riemannian manifold M of dimension n. To this end, we assume that there exists a (possibly infinite) family of smooth vector fields $\{A_i;\ i \in \mathcal{I}\}$ on M satisfying the following conditions:

(a) for all $x \in M$, $\displaystyle\sum_{i\in\mathcal{I}}\langle A_i(x), u\rangle_{T_xM}^2 = |u|_{T_xM}^2$ for any $u \in T_xM$;

(b) $\displaystyle\sum_{i\in\mathcal{I}}\nabla_{A_i}A_i = 0$;

(c) $\displaystyle\sum_{i\in\mathcal{I}}A_i\wedge\nabla_V A_i = 0$ for any vector field V.

Here ∇ denotes the covariant derivative with respect to the Levi–Civita connection on M and $\wedge$ the exterior product. First of all, we give the following example.

Example 3.1 (Gradient system). By Nash's embedding theorem, M can be isometrically embedded into $\mathbb{R}^m$ for some $m > n$. For any $x \in M$, denote by P_x the orthogonal projection from $\mathbb{R}^m$ onto T_xM. Let $e = \{e_1, \cdots, e_m\}$ be an orthonormal basis of $\mathbb{R}^m$. According to [23, Section 4.2], we define

$$A_i(x) = P_x(e_i), \quad x \in M,\ i = 1, \cdots, m.$$

Then $\{A_1, \cdots, A_m\}$ are smooth vector fields satisfying conditions (a), (b) and (c). Note that condition (c) does not often appear. For a justification of (c), we refer to [13, Remark 2.3.1, p.39]. For the case of spheres, we shall do explicit computations in Appendix to illustrate conditions (a), (b) and (c).

Now we shall decompose the de Rham–Hodge Laplacian operator $\square$ as the sum of $\mathcal{L}_{A_i}^2$, where $\mathcal{L}_A$ denotes the Lie derivative with respect to A. For a differential form ω on M, it holds that

$$\mathcal{L}_A d\omega = d\mathcal{L}_A\omega, \tag{3.1}$$

where d is the exterior derivative. Let $I(A)$ be the inner product by A, that is, for a differential q-form ω,

$$(I(A)\omega)(V_2, \cdots, V_q) = \omega(A, V_2, \cdots, V_q).$$

Following [13], we define, for a differential q-form ω,

$$\hat{\delta}(\omega) = \sum_{i \in \mathcal{I}} I(A_i)(\mathcal{L}_{A_i}\omega). \tag{3.2}$$

Let δ be the adjoint operator of d, which admits the expression

$$\delta(\omega)(v_2, \cdots, v_q) = \sum_{j=1}^{n} (\nabla_{u_j}\omega)(u_j, v_2, \cdots, v_q), \tag{3.3}$$

where $\{u_1, \cdots, u_n\}$ is an orthonormal basis of T_xM.

Proposition 3.2. *Under conditions (a) and (b), for any differential* 1*-form* ω*,* $\hat{\delta}(\omega) = \delta(\omega)$*.*

Proof. We have

$$I(A_i)\mathcal{L}_{A_i}\omega = (\mathcal{L}_{A_i}\omega)(A_i) = \mathcal{L}_{A_i}(\omega(A_i)) = \omega(\nabla_{A_i}A_i) + (\nabla_{A_i}\omega)(A_i). \tag{3.4}$$

Let $\{u_1, \cdots, u_n\}$ be an orthonormal basis of T_xM, then condition (a) yields

$$\sum_{i \in \mathcal{I}} \langle A_i(x), u_j \rangle \, \langle A_i(x), u_k \rangle = \langle u_j, u_k \rangle = \delta_{jk}.$$

Therefore, replacing $A_i(x)$ by $\sum_{j=1}^{n} \langle A_i(x), u_j \rangle u_j$ at the last term in (3.4), and summing over $i \in \mathcal{I}$ leads to $\delta(\omega)$ according to (3.3); the sum of the first term on the right hand side of (3.4) vanishes by condition (b). $\square$

Proposition 3.3. *Under (a), (b) and (c), for any differential* 2*-form* ω*,* $\hat{\delta}(\omega) = \delta(\omega)$*.*

Proof. By (3.2), we have

$$\hat{\delta}(\omega)(V) = \sum_{i \in \mathcal{I}} (\mathcal{L}_{A_i}\omega)(A_i, V).$$

Next,

$$\begin{aligned}
(\mathcal{L}_{A_i}\omega)(A_i, V) &= \mathcal{L}_{A_i}\big(\omega(A_i, V)\big) - \omega(A_i, \mathcal{L}_{A_i}V) \\
&= (\nabla_{A_i}\omega)(A_i, V) + \omega(\nabla_{A_i}A_i, V) + \omega(A_i, \nabla_{A_i}V) - \omega(A_i, \mathcal{L}_{A_i}V) \\
&= (\nabla_{A_i}\omega)(A_i, V) + \omega(\nabla_{A_i}A_i, V) + \omega(A_i, \nabla_V A_i),
\end{aligned}$$

since $\nabla_{A_i}V - \nabla_V A_i = \mathcal{L}_{A_i}V$. By condition (c), $\sum_{i \in \mathcal{I}} \omega(A_i, \nabla_V A_i) = 0$. Summing over $i \in \mathcal{I}$ and according to (b) and (3.3), we get the result. $\square$

Now the de Rham–Hodge Laplacian operator $\square = d\delta + \delta d$ admits the following decomposition (see [13]):

Theorem 3.4. *Under the conditions (a)–(c), for any differential* 1*-form* ω*, we have*

$$\sum_{i \in \mathcal{I}} \mathcal{L}_{A_i}^2 \omega = \square\omega. \tag{3.5}$$

Proof. Applying Cartan's formula $\mathcal{L}_{A_i}\omega = I(A_i)d\omega + dI(A_i)\omega$ to $\mathcal{L}_{A_i}\omega$, we have

$$\begin{aligned}
\mathcal{L}_{A_i}^2 \omega &= I(A_i)d\mathcal{L}_{A_i}\omega + dI(A_i)\mathcal{L}_{A_i}\omega \\
&= I(A_i)\mathcal{L}_{A_i}(d\omega) + dI(A_i)\mathcal{L}_{A_i}\omega,
\end{aligned}$$

where we used (3.1) for the second equality. Now by Propositions 3.2 and 3.3, we get

$$\sum_{i \in \mathcal{I}} \mathcal{L}_{A_i}^2 \omega = \delta d\omega + d\delta\omega = \square\omega.$$

The theorem is proved. $\square$

Recall that on a Riemannian manifold, there is a one-to-one correspondence between the space of vector fields and that of differential 1-forms. Given a vector field A (resp. differential 1-form θ), we shall denote by A^* (resp. $\theta^\#$) the corresponding differential 1-form (resp. vector field). The action of the de Rham–Hodge Laplacian $\square$ on the vector field A is defined as follows:

$$\square A := (\square A^*)^\#. \tag{3.6}$$

Lemma 3.5. *The conditions (b) and (c) imply*

$$\sum_{i\in\mathcal{I}} \mathrm{div}(A_i)A_i = 0. \tag{3.7}$$

Proof. We have $I(V)(A_i \wedge \nabla_V A_i) = \langle A_i, V\rangle \nabla_V A_i - \langle \nabla_V A_i, V\rangle A_i$. Let $\{v_1, \cdots, v_n\}$ be an orthonormal basis, then by condition (c),

$$\begin{aligned} 0 &= \sum_{i\in\mathcal{I}}\sum_{j=1}^{n}\big(\langle A_i, v_j\rangle \nabla_{v_j} A_i - \langle \nabla_{v_j} A_i, v_j\rangle A_i\big) \\ &= \sum_{i\in\mathcal{I}} \nabla_{A_i} A_i - \sum_{i\in\mathcal{I}} \mathrm{div}(A_i)A_i. \end{aligned}$$

The first term vanishes by condition (b); therefore (3.7) follows. $\square$

Remark 3.6. *When the manifold M is embedded in some $\mathbb{R}^N$, the relation* (3.7) *was proved in [23, p.102]. However, in order to prove the next result, the equality* (3.7) *is* not sufficient*, we have to assume the following condition:*

(d) $\sum_{i\in\mathcal{I}} \mathrm{div}(A_i)\mathcal{L}_{A_i} = 0.$

Unfortunately the vector fields $\{A_1, \cdots, A_m\}$ in Example 3.1 do not satisfy condition (d). See the appendix.

Theorem 3.7. *Under (a), (b), (c) and (d), we have, for any vector field B,*

$$\square B = \sum_{i\in\mathcal{I}} \mathcal{L}_{A_i}^2 B. \tag{3.8}$$

Proof. Let ω be a differential 1-form. We have

$$\mathcal{L}_{A_i}(\omega(B)) = (\mathcal{L}_{A_i}\omega)(B) + \omega(\mathcal{L}_{A_i}B),$$

and

$$\mathcal{L}_{A_i}^2(\omega(B)) = (\mathcal{L}_{A_i}^2\omega)(B) + \omega(\mathcal{L}_{A_i}^2 B) + 2(\mathcal{L}_{A_i}\omega)(\mathcal{L}_{A_i}B).$$

By the integration by parts formula,

$$\begin{aligned} \int_M (\mathcal{L}_{A_i}\omega)(\mathcal{L}_{A_i}B)\,\mathrm{d}x &= \int_M \mathcal{L}_{A_i}(\omega(\mathcal{L}_{A_i}B))\,\mathrm{d}x - \int_M \omega(\mathcal{L}_{A_i}^2 B)\,\mathrm{d}x \\ &= -\int_M \mathrm{div}(A_i)\omega(\mathcal{L}_{A_i}B)\,\mathrm{d}x - \int_M \omega(\mathcal{L}_{A_i}^2 B)\,\mathrm{d}x. \end{aligned}$$

Therefore,

$$\int_M \mathcal{L}_{A_i}^2(\omega(B))\,\mathrm{d}x = \int_M (\mathcal{L}_{A_i}^2\omega)(B)\,\mathrm{d}x - \int_M \omega(\mathcal{L}_{A_i}^2 B)\,\mathrm{d}x - 2\int_M \mathrm{div}(A_i)\omega(\mathcal{L}_{A_i}B)\,\mathrm{d}x.$$

By condition (d), $\sum_{i\in\mathcal{I}}\int_M \mathrm{div}(A_i)\omega(\mathcal{L}_{A_i}B)\,\mathrm{d}x = 0$. If we denote by $\hat{\Box}B = \sum_{i\in\mathcal{I}}\mathcal{L}_{A_i}^2 B$, then summing over i and according to (3.5), we get

$$\int_M \Delta(\omega(B))\,\mathrm{d}x = \int_M (\Box\omega)(B)\,\mathrm{d}x - \int_M \omega(\hat{\Box}B)\,\mathrm{d}x.$$

It follows that $\int_M (\Box\omega)(B)\,\mathrm{d}x = \int_M \omega(\hat{\Box}B)\,\mathrm{d}x$; therefore $\Box B = \hat{\Box}B$. □

Proposition 3.8. *If* $\mathrm{div}(B) = 0$, *then* $\mathrm{div}(\Box B) = 0$.

Proof. Notice first that $\delta(B^*) = \mathrm{div}(B) = 0$, then by (3.6),

$$\mathrm{div}(\Box B) = \delta(\Box B^*) = \delta d\delta(B^*) = 0,$$

which completes the proof. □

In what follows, we consider the vector fields $\{A_i;\, i\in\mathcal{I}\}$ which satisfy the conditions (a)–(c). Let $W_t = \{W_t^i;\, i\in\mathcal{I}\}$ be a family of independent standard Brownian motions; consider the Stratonovich SDE on M:

$$\mathrm{d}X_t = \sum_{i\in\mathcal{I}} A_i(X_t)\circ \mathrm{d}W_t^i + u_t(X_t)\,\mathrm{d}t, \quad X_0 = x\in M. \tag{3.9}$$

Assume that $u_t\in C^{1,\alpha}$, then X_t is a stochastic flow of C^1-diffeomorphisms of M. Let

$$\mathrm{d}\big[(X_t)_\#(\mathrm{d}x)\big] = \rho_t\,\mathrm{d}x, \quad \mathrm{d}\big[(X_t^{-1})_\#(\mathrm{d}x)\big] = \tilde{\rho}_t\,\mathrm{d}x,$$

where $(X_t)_\#(\mathrm{d}x)$ means the push-forward measure of $\mathrm{d}x$ by X_t. By [18, Lemma 4.3.1], $\tilde{\rho}$ admits the expression

$$\tilde{\rho}_t(x) = \exp\bigg\{-\sum_{i\in\mathcal{I}}\int_0^t \mathrm{div}(A_i)(X_s(x))\circ\mathrm{d}W_s^i - \int_0^t \mathrm{div}(u_s)(X_s(x))\,\mathrm{d}s\bigg\}. \tag{3.10}$$

Since for any $f\in C(M)$, it holds

$$\int_M f(x)\,\mathrm{d}x = \int_M f\big(X_t^{-1}(X_t)\big)\,\mathrm{d}x = \int_M f(X_t^{-1})\rho_t\,\mathrm{d}x = \int_M f\,\rho_t(X_t)\tilde{\rho}_t\,\mathrm{d}x,$$

we have

$$\rho_t(X_t)\tilde{\rho}_t = 1. \tag{3.11}$$

Before stating the main result of this work, we introduce a notation. Let $f: M\to M$ be a C^1-map, then for each $x\in M$, $df(x): T_xM\to T_{f(x)}M$. We define $(df)^*(x): T_{f(x)}M\to T_xM$ by

$$\langle (df)^*(x)v, u\rangle_{T_xM} = \langle df(x)u, v\rangle_{T_{f(x)}M}, \quad u\in T_xM, v\in T_{f(x)}M.$$

Let ω be a differential 1-form on M, the pull-back $f^*\omega$ of ω by f is defined by

$$\langle f^*\omega, v\rangle_x = \langle \omega_{f(x)}, df(x)v\rangle.$$

Then if f is a diffeomorphism,

$$\langle f^*\omega, v\rangle_x = \langle \omega, f_*v\rangle_{f(x)}.$$

Theorem 3.9 (Stochastic Lagrangian representation)**.** *Let M be a compact Riemannian manifold such that there is a family of vector fields $\{A_i;\, i\in\mathcal{I}\}$ satisfying the conditions:*

(a) *for all* $x \in M$, $\sum_{i\geq 1}\langle A_i(x), u\rangle^2_{T_xM} = |u|^2_{T_xM}$ *for any* $u \in T_xM$;

(b) $\sum_{i\geq 1} \nabla_{A_i} A_i = 0$;

(c) $\sum_{i\geq 1} A_i \wedge \nabla_V A_i = 0$ *for any vector field* V;

(d) $\sum_{i\geq 1} \mathrm{div}(A_i)\mathcal{L}_{A_i} = 0$ *holds on vector fields.*

Let $\nu > 0$ *and* u_0 *be a divergence-free vector field on* M. *Assume that* $u_t \in C^{2,\alpha}$. *Then the pair* (X, u) *satisfies*

$$\begin{cases} \mathrm{d}X_t = \sqrt{2\nu}\,\sum_{i=1}^m A_i(X_t)\circ \mathrm{d}W_t^i + u_t(X_t)\,\mathrm{d}t, \quad X_0 = x, \\ u_t = \mathbb{E}\mathbf{P}\Big[\rho_t\,(dX_t^{-1})^*\cdot u_0(X_t^{-1})\Big], \end{cases} \tag{3.12}$$

if and only if u *solves the Navier–Stokes equations on* M:

$$\begin{cases} \partial_t u + \nabla_u u - \nu\Box u + \nabla p = 0, \\ \mathrm{div}(u) = 0, \quad u|_{t=0} = u_0. \end{cases} \tag{3.13}$$

Moreover, u_t *has the following more geometric expression*

$$u_t = \mathbb{E}\Big[\mathbf{P}\big(\rho_t\,(X_t^{-1})^* u_0^*\big)^{\#}\Big]. \tag{3.14}$$

Proof. Let v be a divergence-free vector field on M. We have

$$\begin{aligned} \int_M \langle u_t, v\rangle\,\mathrm{d}x &= \mathbb{E}\int_M \rho_t\,\big\langle (dX_t^{-1})^*\cdot u_0(X_t^{-1}),\, v\big\rangle\,\mathrm{d}x \\ &= \mathbb{E}\int_M \rho_t\,\big\langle (dX_t^{-1})\,v,\, u_0(X_t^{-1})\big\rangle\,\mathrm{d}x \\ &= \mathbb{E}\int_M \rho_t(X_t)\,\tilde\rho_t\,\big\langle dX_t^{-1}(X_t)v(X_t),\, u_0\big\rangle\,\mathrm{d}x. \end{aligned}$$

Now using (2.1) and (3.11), we get the following expression, similar to (2.2):

$$\int_M \langle u_t, v\rangle\,\mathrm{d}x = \mathbb{E}\bigg(\int_M \big\langle u_0, X_{t*}^{-1}(v)\big\rangle\,\mathrm{d}x\bigg). \tag{3.15}$$

Again by [17, p.265, Theorem 2.1] and (3.8), we have

$$\begin{aligned} X_{t*}^{-1}(v) &= v + \sum_{i=1}^m \int_0^t X_{s*}^{-1}(\mathcal{L}_{A_i}v)\,\mathrm{d}W_s^i + \nu\sum_{i=1}^m\int_0^t X_{s*}^{-1}(\mathcal{L}_{A_i}^2 v)\,\mathrm{d}s + \int_0^t X_{s*}^{-1}(\mathcal{L}_{u_s}v)\mathrm{d}s \\ &= v + \sum_{i=1}^m \int_0^t X_{s*}^{-1}(\mathcal{L}_{A_i}v)\,\mathrm{d}W_s^i + \nu\int_0^t X_{s*}^{-1}(\Box v)\,\mathrm{d}s + \int_0^t X_{s*}^{-1}(\mathcal{L}_{u_s}v)\mathrm{d}s. \end{aligned}$$

Substituting $X_{t*}^{-1}(v)$ into (3.15), we have

$$\begin{aligned} \int_M \langle u_t, v\rangle\,\mathrm{d}x = \int_M \langle u_0, v\rangle\,\mathrm{d}x &+ \nu\int_0^t \mathbb{E}\bigg(\int_M \big\langle u_0, X_{s*}^{-1}(\Box v)\big\rangle\,\mathrm{d}x\bigg)\mathrm{d}s \\ &+ \int_0^t \mathbb{E}\bigg(\int_M \big\langle u_0,\, X_{s*}^{-1}(\mathcal{L}_{u_s}v)\big\rangle\,\mathrm{d}x\bigg)\mathrm{d}s. \end{aligned}$$

Now by Lemma 2.1 and Proposition 3.8, $\mathcal{L}_{u_s}v$ and $\Box v$ are of divergence free. Substituting respectively v in (3.15) by $\mathcal{L}_{u_s}v$ and $\Box v$ yields

$$\int_M \langle u_t, v\rangle \,\mathrm{d}x = \int_M \langle u_0, v\rangle \,\mathrm{d}x + \nu \int_0^t\int_M \langle u_s, \Box v\rangle \,\mathrm{d}x\mathrm{d}s + \int_0^t\int_M \langle u_s, \mathcal{L}_{u_s}v\rangle \,\mathrm{d}x\mathrm{d}s. \tag{3.16}$$

Since M is torsion-free, we have $\mathcal{L}_{u_s}v = [u_s, v] = \nabla_{u_s}v - \nabla_v u_s$. As a result,

$$\begin{aligned}\int_M \langle u_s, \mathcal{L}_{u_s}v\rangle \,\mathrm{d}x &= \int_M \langle u_s, \nabla_{u_s}v\rangle \,\mathrm{d}x - \int_M \langle u_s, \nabla_v u_s\rangle \,\mathrm{d}x \\ &= \int_M \langle u_s, \nabla_{u_s}v\rangle \,\mathrm{d}x - \frac{1}{2}\int_M v(|u_s|^2) \,\mathrm{d}x = \int_M \langle u_s, \nabla_{u_s}v\rangle \,\mathrm{d}x. \end{aligned}\tag{3.17}$$

By (3.16) and (3.17), we know that for a.e. $t \geq 0$, it holds

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_M \langle u_t, v\rangle \,\mathrm{d}x = \nu \int_M \langle u_t, \Box v\rangle \,\mathrm{d}x + \int_M \langle u_t, \nabla_{u_t}v\rangle \,\mathrm{d}x.$$

Multiplying both sides by $\alpha \in C_c^1([0,\infty))$ and integrating by parts on $[0,\infty)$, we arrive at

$$\alpha(0)\int_M \langle u_0, v\rangle \,\mathrm{d}x + \int_0^\infty \int_M \big[\alpha'(t)\langle u_t, v\rangle + \alpha(t)\langle u_t, \nabla_{u_t}v\rangle + \nu\,\alpha(t)\langle u_t, \Box v\rangle\big]\mathrm{d}x\mathrm{d}t = 0.$$

The above equation is the weak formulation of the Navier–Stokes (3.13) on the manifold M. Since $u_t \in C^{2,\alpha}$, it is a strong solution to (3.13).

For proving the converse, we use the idea in [25, Theorem 2.3]. Let $u_t \in C^{2,\alpha}$ be a solution to (3.13), then

$$\int_M \langle u_t, v\rangle \,\mathrm{d}x = \int_M \langle u_0, v\rangle \,\mathrm{d}x + \nu \int_0^t\int_M \langle u_s, \Box v\rangle \,\mathrm{d}x\mathrm{d}s + \int_0^t\int_M \langle u_s, \mathcal{L}_{u_s}v\rangle \,\mathrm{d}x\mathrm{d}s.$$

Consider the SDE in (3.12) with drift term u_t. Define

$$\tilde u_t = \mathbb{E}\mathbf{P}\Big[\rho_t\,(dX_t^{-1})^* \cdot u_0(X_t^{-1})\Big].$$

Then the same calculation as above leads to

$$\int_M \langle \tilde u_t, v\rangle \,\mathrm{d}x = \int_M \langle u_0, v\rangle \,\mathrm{d}x + \nu \int_0^t\int_M \langle \tilde u_s, \Box v\rangle \,\mathrm{d}x\mathrm{d}s + \int_0^t\int_M \langle \tilde u_s, \mathcal{L}_{u_s}v\rangle \,\mathrm{d}x\mathrm{d}s.$$

Let $z_t = u_t - \tilde u_t$; we have

$$\int_M \langle z_t, v\rangle \,\mathrm{d}x = \nu \int_0^t\int_M \langle z_s, \Box v\rangle \,\mathrm{d}x\mathrm{d}s + \int_0^t\int_M \langle z_s, \mathcal{L}_{u_s}v\rangle \,\mathrm{d}x\mathrm{d}s.$$

It follows that (z_t) solves the following heat equation on M

$$\frac{\mathrm{d}z_t}{\mathrm{d}t} = \nu\,\Box z_t - \mathcal{L}_{u_t}z_t, \quad z_0 = 0.$$

By uniqueness of solutions, we get that $z_t = 0$ for all $t \geq 0$. Thus $u_t = \tilde u_t$.

To prove (3.14), we note that

$$\begin{aligned}\int_M \rho_t\,\big\langle (X_t^{-1})^* u_0^*, v\big\rangle \,\mathrm{d}x &= \int_M \rho_t\,\big\langle u_0^*, (X_t^{-1})_* v\big\rangle_{X_t^{-1}} \,\mathrm{d}x \\ &= \int_M \rho_t(X_t)\,\tilde\rho_t\,\big\langle u_0^*, (X_t^{-1})_* v\big\rangle \,\mathrm{d}x \\ &= \int_M \big\langle u_0^*, (X_t^{-1})_* v\big\rangle \,\mathrm{d}x = \int_M \big\langle u_0, (X_t^{-1})_* v\big\rangle_{T_xM} \,\mathrm{d}x\end{aligned}$$

Now by (3.15), we have, for any vector field v of divergence free,

$$\int_M \langle u_t, v\rangle \,\mathrm{d}x = \mathbb{E}\Big(\int_M \rho_t \,\langle (X_t^{-1})^* u_0^*, v\rangle \,\mathrm{d}x\Big).$$

Then (3.14) follows. The proof of Theorem 3.9 is complete. □

4 Volume-preserving flows on the torus and the sphere

It is usually difficult to find on a general Riemannian manifold a family of vector fields of *divergence free*, which satisfy the conditions (a)–(d) in Section 3. In this part, we shall treat two important examples: torus $\mathbb{T}^n$ and sphere $\mathbb{S}^n$ in which this is possible.

4.1 Case of torus $\mathbb{T}^2$

For the simplicity of exposition, we only consider the two dimensional torus $\mathbb{T}^2$. Let $\mathbb{Z}^2$ be the set of lattice points in $\mathbb{R}^2$ and define $\mathbb{Z}_0^2 = \mathbb{Z}^2 \setminus \{(0,0)^*\}$, where $*$ means the column vector. For $k \in \mathbb{Z}_0^2$, we define $k^\perp = (k_2, -k_1)^*$ and

$$A_k(\theta) = \frac{\cos(k\cdot\theta)}{|k|^\beta}\cdot\frac{k^\perp}{|k|}, \quad B_k(\theta) = \frac{\sin(k\cdot\theta)}{|k|^\beta}\cdot\frac{k^\perp}{|k|}, \quad \theta\in\mathbb{T}^2, \tag{4.1}$$

where $\beta > 1$ is some constant. Then the family $\{A_k, B_k : k \in \mathbb{Z}_0^2\}$ constitutes an orthogonal basis of the space of divergence free vector fields V on $\mathbb{T}^2$ such that $\int_{\mathbb{T}^2} V\,\mathrm{d}\theta = 0$ (see [10]). We shall show that the family $\{A_k, B_k;\ k \in \mathbb{Z}_0^2\}$ of vector fields satisfy the conditions (a)–(c).

Firstly, for any $u \in \mathbb{R}^2$, we remark that

$$\langle A_k(\theta), u\rangle^2 + \langle B_k(\theta), u\rangle^2 = \frac{\langle k^\perp, u\rangle^2}{|k|^{2(\beta+1)}}\big(\cos^2(k\cdot\theta) + \sin^2(k\cdot\theta)\big) = \frac{\langle k^\perp, u\rangle^2}{|k|^{2(\beta+1)}}.$$

Thus

$$\sum_{k\in\mathbb{Z}_0^2}\big(\langle A_k(\theta), u\rangle^2 + \langle B_k(\theta), u\rangle^2\big) = \sum_{k\in\mathbb{Z}_0^2}\frac{\langle k^\perp, u\rangle^2}{|k|^{2(\beta+1)}}.$$

Notice that the transform $k \mapsto k^\perp$ on $\mathbb{Z}_0^2$ is one-to-one and preserves the Euclidean norm $|\cdot|$, we have

$$\sum_{k\in\mathbb{Z}_0^2}\frac{\langle k^\perp, u\rangle^2}{|k|^{2(\beta+1)}} = \sum_{k\in\mathbb{Z}_0^2}\frac{\langle k, u\rangle^2}{|k|^{2(\beta+1)}}$$

and $|u|^2\,|k|^2 = \langle k, u\rangle^2 + \langle k^\perp, u\rangle^2$, therefore

$$\sum_{k\in\mathbb{Z}_0^2}\frac{|u|^2}{|k|^{2\beta}} = 2\sum_{k\in\mathbb{Z}_0^2}\frac{\langle k^\perp, u\rangle^2}{|k|^{2(\beta+1)}}.$$

Consequently,

$$\sum_{k\in\mathbb{Z}_0^2}\big(\langle A_k(\theta), u\rangle^2 + \langle B_k(\theta), u\rangle^2\big) = \frac{|u|^2}{2}\sum_{k\in\mathbb{Z}_0^2}\frac{1}{|k|^{2\beta}} = \nu_0\,|u|^2, \tag{4.2}$$

where

$$\nu_0 = \frac{1}{2}\sum_{k\in\mathbb{Z}_0^2}\frac{1}{|k|^{2\beta}}.$$

Therefore (4.2) implies that $\{\frac{A_k}{\sqrt{\nu_0}}, \frac{B_k}{\sqrt{\nu_0}},\ k \in \mathbb{Z}_0^2\}$ satisfies the condition (a).

Secondly, by the definition of A_k,

$$\nabla_{A_k} A_k = \frac{k^\perp}{|k|^{\beta+1}} \langle A_k, \nabla_\theta \cos(k\cdot\theta)\rangle = -\frac{k^\perp}{|k|^{2(\beta+2)}} \cos(k\cdot\theta)\sin(k\cdot\theta)\langle k^\perp, k\rangle = 0.$$

Similarly, $\nabla_{B_k} B_k = 0$. Therefore condition (b) is satisfied. Finally, for any vector field V on $\mathbb{T}^2$, we have

$$\nabla_V A_k = \frac{k^\perp}{|k|^{\beta+1}} \langle V, \nabla_\theta \cos(k\cdot\theta)\rangle = -\frac{k^\perp}{|k|^{\beta+1}} \sin(k\cdot\theta)\langle V, k\rangle.$$

In the same way,

$$\nabla_V B_k = \frac{k^\perp}{|k|^{\beta+1}} \cos(k\cdot\theta)\langle V, k\rangle.$$

Hence, for any $u_1, u_2 \in \mathbb{R}^2$, it holds that

$$\begin{aligned}
&\langle A_k, u_1\rangle\langle \nabla_V A_k, u_2\rangle + \langle B_k, u_1\rangle\langle \nabla_V B_k, u_2\rangle \\
&= -\frac{\langle V, k\rangle \sin(k\cdot\theta)\cos(k\cdot\theta)}{|k|^{2(\beta+1)}}\langle k^\perp, u_1\rangle\langle k^\perp, u_2\rangle + \frac{\langle V, k\rangle \sin(k\cdot\theta)\cos(k\cdot\theta)}{|k|^{2(\beta+1)}}\langle k^\perp, u_1\rangle\langle k^\perp, u_2\rangle \\
&= 0,
\end{aligned}$$

that is, the condition (c) is also satisfied.

Now let $\{u_t;\ t \geq 0\}$ be a family of $C^{2,\alpha}$-vector fields of divergence free on $\mathbb{T}^2$. Consider the following SDE

$$\mathrm{d}X_t = \sqrt{\frac{2\nu}{\nu_0}} \sum_{k\in\mathbb{Z}_0^2} \big(A_k(X_t)\circ \mathrm{d}W_t^k + B_k(X_t)\circ \mathrm{d}\tilde{W}_t^k\big) + u_t(X_t)\,\mathrm{d}t, \quad X_0 = x \in \mathbb{T}^2, \tag{4.3}$$

where $\{W_t^k, \tilde{W}_t^k;\ k \in \mathbb{Z}_0^2\}$ is a family of independent standard real Brownian motions. When $\beta > 3$, the SDE (4.3) defines a flow of C^1-diffeomorphisms of $\mathbb{T}^2$ (see [10]). In this case, by (3.10), for almost surely w, $x \to X_t(x, w)$ preserves the measure $\mathrm{d}x$; therefore by Theorem 3.9, we have

Theorem 4.1. *$u_t \in C^{2,\alpha}$ with initial value u_0 is a solution of the Navier–Stokes equations on $\mathbb{T}^2$ if and only if*

$$u_t = \mathbb{E}\Big[\mathbf{P}\big((X_t^{-1})^* u_0^*\big)^{\#}\Big]. \tag{4.4}$$

4.2 Case of sphere $\mathbb{S}^n$

Let $\square$ be the de Rham–Hodge Laplacian operator acting on vector fields over $\mathbb{S}^n$. For $\ell \geq 1$, set $c_{\ell,\delta} = (\ell+1)(\ell+n-2)$. Then $\{c_{\ell,\delta}; \ell \geq 1\}$ are the eigenvalues of $\square$ corresponding to the divergence free eigenvector fields. Denote by $\mathcal{D}_\ell$ the eigenspace associated to $c_{\ell,\delta}$ and $d_\ell = \dim(\mathcal{D}_\ell)$ the dimension of $\mathcal{D}_\ell$. It is known that

$$d_\ell \sim O(\ell^{n-1}) \quad \text{as} \quad \ell \to +\infty.$$

For $\ell \geq 1$, let $\{V_{\ell,k}; k = 1, \ldots, d_\ell\}$ be an orthonormal basis of $\mathcal{D}_\ell$ in L^2:

$$\int_{\mathbb{S}^n} \big\langle V_{\ell,k}(x), V_{\alpha,\beta}(x)\big\rangle\,\mathrm{d}x = \delta_{\ell\alpha}\delta_{k\beta}.$$

Weyl's theorem implies that the vector fields $\{V_{\ell,k}; k = 1, \ldots, d_\ell, \ \ell \geq 1\}$ are smooth. We refer to [22] for a detailed study on isotropic flows on $\mathbb{S}^n$, many properties below were proved there. But we are more familiar with [16] to which we refer known results. Let $\{b_\ell; \ell \geq 1\}$ be a family of positive numbers such that $\sum_{\ell=1}^{\infty} b_\ell < +\infty$. Set

$$A_{\ell,k} = \sqrt{\frac{nb_\ell}{d_\ell}}\, V_{\ell,k}.$$

Below we shall consider the family

$$\big\{A_{\ell,k};\ 1 \leq k \leq d_\ell, \ell \geq 1\big\}.$$

Let's first check the condition (a). By [16, (A.13)], we have, for $x, y \in \mathbb{S}^n$

$$\frac{n}{d_\ell} \sum_{k=1}^{d_\ell} \langle V_{\ell,k}(x), y\rangle^2 = \sin^2\theta, \tag{4.5}$$

where θ is the angle between x and y. Let $u \in T_x\mathbb{S}^n$; then $\langle x, u\rangle = 0$. By (4.5),

$$\frac{n}{d_\ell} \sum_{k=1}^{d_\ell} \langle V_{\ell,k}(x), u\rangle^2 = |u|^2.$$

Therefore,

$$\sum_{\ell\geq 1} \sum_{k=1}^{d_\ell} \langle A_{\ell,k}(x), u\rangle^2 = \sum_{\ell \geq 1} \frac{nb_\ell}{d_\ell} \sum_{k=1}^{d_\ell} \langle V_{\ell,k}(x), u\rangle^2 = \nu_0\, |u|^2,$$

where

$$\nu_0 = \sum_{\ell\geq 1} b_\ell.$$

Next, by [16, Propositions A.3 and A.5],

$$\sum_{k=1}^{d_\ell} \nabla_{V_{\ell,k}} V_{\ell,k} = 0. \tag{4.6}$$

thus the condition (b) is satisfied.

It remains to check the condition (c). To this end, we need a bit more description on $V_{\ell,k}$. Let $\{e_1, \cdots, e_{n+1}\}$ be the canonical basis of $\mathbb{R}^{n+1}$. We denote by $P_0 = e_{n+1}$ the north pole. When $n \geq 3$, the group $SO(n+1)$ acts transitively on $\mathbb{S}^n$. Let $x \in \mathbb{S}^n$ be fixed, then there is $g \in SO(n+1)$ such that $x = \chi_g(P_0) = gP_0$. Then

$$V_{\ell,k}(gP_0) = \sqrt{\frac{d_\ell}{n}} \sum_{i=1}^{n} Q^\ell_{ki}(g) d\chi_g(P_0) e_i, \tag{4.7}$$

where $\{Q^\ell;\ \ell \geq 1\}$ is the family of irreducible unitary representations of $SO(n+1)$ which keep the representation $h \to d\chi_h(P_0)$. It is important that the element Q^ℓ_{qi} has an explicit formula for $1 \leq q, i \leq n$:

$$Q^\ell_{qi}(g) = \Big(t\gamma_\ell(t) - \frac{1-t^2}{n-1}\gamma'_\ell(t)\Big) g_{qi} - \Big(\gamma_\ell(t) + \frac{t}{n-1}\gamma'_\ell(t)\Big)\, g_{q,n+1} g_{n+1,i}, \tag{4.8}$$

with $t = g_{n+1,n+1}$ and

$$\gamma_\ell(\cos\theta) = \int_0^\pi \big(\cos\theta - \sqrt{-1}\sin\theta\,\cos\varphi\big)^{\ell-1}\,\sin^n\varphi\,\frac{\mathrm{d}\varphi}{c_n},$$

where $c_n = \int_0^\pi \sin^n\varphi\,\mathrm{d}\varphi$. Set $E_j = d\chi_g(P_0)e_j$; then $\{E_1,\cdots,E_n\}$ is an orthonormal basis of $T_x\mathbb{S}^n$. Fix j, we consider $\hat{g}(s) \in SO(n+1)$ which leaves invariant e_i for $i \neq j$, $i \neq n+1$ and

$$\begin{cases} \hat{g}(s)e_j &= \cos s\, e_j - \sin s\, e_{n+1}, \\ \hat{g}(s)e_{n+1} &= \sin s\, e_j + \cos s\, e_{n+1}. \end{cases}$$

Then by [16, p.596],

$$\nabla_{E_j} V_{\ell,k}(x) = \sqrt{\frac{d_\ell}{n}} \sum_{i=1}^n \sum_{\beta=1}^{d_\ell} Q^\ell_{k\beta}(g)\Big\{\frac{\mathrm{d}}{\mathrm{d}s}\Big|_{s=0} Q^\ell_{\beta i}(\hat{g}(s))\Big\}E_i. \tag{4.9}$$

Combining (4.7) and (4.9), we get

$$\begin{aligned}\sum_{k=1}^{d_\ell} V_{\ell,k}\wedge\nabla_{E_j}V_{\ell,k} &= \frac{d_\ell}{n}\sum_{q,i=1}^{n}\sum_{\beta,k=1}^{d_\ell} Q^\ell_{k\beta}Q^\ell_{kq}\Big\{\frac{\mathrm{d}}{\mathrm{d}s}\Big|_{s=0}Q^\ell_{\beta i}(\hat{g}(s))\Big\}E_q\wedge E_i \\ &= \frac{d_\ell}{n}\sum_{q,i=1}^{n}\Big\{\frac{\mathrm{d}}{\mathrm{d}s}\Big|_{s=0}Q^\ell_{qi}(\hat{g}(s))\Big\}E_q\wedge E_i.\end{aligned}$$

In (4.8), we replace g by $\hat{g}(s)$; therefore $t = \cos s$, the term $g_{qi} = 0$ for $q \neq i$, $g_{i,n+1} = 0$ if $i \neq j$, $g_{n+1,i} = 0$ if $i \neq j$. We have $g_{jj} = \cos s$ and $g_{n+1,j}g_{j,n+1} = -\sin^2 s$. It follows that

$$\sum_{k=1}^{d_\ell} V_{\ell,k}\wedge\nabla_{E_j}V_{\ell,k} = 0.$$

The condition (c) is satisfied. Notice that using (4.8), we have in fact the stronger result

$$\sum_{k=1}^{d_\ell} V_{\ell,k}\otimes\nabla_{E_j}V_{\ell,k} = 0.$$

Now let $\{u_t;\, t \geq 0\}$ be a family of $C^{2,\alpha}$-vector fields of divergence free on $\mathbb{S}^n$. Let $b_\ell = 1/\ell^{1+\alpha}$. Consider the following SDE

$$\mathrm{d}X_t = \sqrt{\frac{2\nu}{\nu_0}}\sum_{\ell\geq 1}\sum_{k=1}^{d_\ell} A_{\ell,k}(X_t)\circ\mathrm{d}W_t^{\ell,k} + u_t(X_t)\,\mathrm{d}t, \quad X_0 = x \in \mathbb{S}^n, \tag{4.10}$$

where $\big\{W_t^{\ell,k};\ \ell \geq 1, 1 \leq k \leq d_\ell\big\}$ is a family of independent standard real Brownian motions. When $\alpha > 2$, the SDE (4.10) defines a flow of C^1-diffeomorphisms of $\mathbb{S}^n$ (see [19, 21]). In this case, for almost surely w, $x \to X_t(x,w)$ preserves the measure $\mathrm{d}x$; therefore by Theorem 3.9, we have

Theorem 4.2. *The velocity $u_t \in C^{2,\alpha}$ with initial value u_0 is a solution of the Navier–Stokes equation on $\mathbb{S}^n$ if and only if*

$$u_t = \mathbb{E}\Big[\mathbf{P}\big((X_t^{-1})^* u_0^*\big)^{\#}\Big]. \tag{4.11}$$

5 Appendix: example of the sphere

For reader's convenience, we shall exhibit properties (a)–(c) in Section 3 in the case of sphere $\mathbb{S}^n$. We denote by $\langle\, ,\rangle$ the canonical inner product of $\mathbb{R}^{n+1}$. Let $x \in \mathbb{S}^n$, the tangent space $T_x\mathbb{S}^n$ of $\mathbb{S}^n$ at the point x is given by

$$T_x\mathbb{S}^n = \big\{v \in \mathbb{R}^{n+1};\ \langle v, x\rangle = 0\big\}.$$

Then the orthogonal projection $P_x : \mathbb{R}^{n+1} \to T_x\mathbb{S}^n$ has the expression:

$$P_x(y) = y - \langle x, y\rangle x.$$

Let $\{e_1, \cdots, e_{n+1}\}$ be an orthonormal basis of $\mathbb{R}^{n+1}$; then the vector fields $A_i(x) = P_x(e_i)$ have the expression: $A_i(x) = e_i - \langle x, e_i\rangle\, x$ for $i = 1, \cdots, n+1$. Let $v \in T_x\mathbb{S}^n$ such that $|v| = 1$, consider

$$\gamma(t) = x\,\cos t + v\,\sin t.$$

Then $\{\gamma(t);\ t \in [0,1]\}$ is the geodesic on $\mathbb{S}^n$ such that $\gamma(0) = x, \gamma'(0) = v$. We have $A_i(\gamma(t)) = e_i - \langle\gamma(t), e_i\rangle\,\gamma(t)$. Taking the derivative with respect to t and at $t = 0$, we get

$$(\nabla_v A_i)(x) = P_x\big(-\langle v, e_i\rangle x - \langle x, e_i\rangle v\big) = -\langle x, e_i\rangle v. \tag{5.1}$$

It follows that

$$\mathrm{div}(A_i) = -n\langle x, e_i\rangle. \tag{5.2}$$

Hence,

$$\sum_{i=1}^{n+1} \mathrm{div}(A_i)A_i = -n\sum_{i=1}^{n+1}\big(\langle x, e_i\rangle e_i - \langle x, e_i\rangle^2 x\big) = -n(x - x) = 0. \tag{5.3}$$

Replacing v by A_i in (5.1), we have $\nabla_{A_i}A_i = -\langle x, e_i\rangle e_i + \langle x, e_i\rangle^2 x$; therefore summing over i, we get

$$\sum_{i=1}^{n+1} \nabla_{A_i}A_i = 0. \tag{5.4}$$

Now let $v \in T_x\mathbb{S}^n$ and $a, b \in T_x\mathbb{S}^n$, we have

$$\begin{aligned}\langle A_i \wedge \nabla_v A_i, a \wedge b\rangle &= \langle A_i, a\rangle\langle\nabla_v A_i, b\rangle - \langle A_i, b\rangle\langle\nabla_v A_i, a\rangle \\ &= \langle a, e_i\rangle\langle x, e_i\rangle\langle v, b\rangle - \langle x, e_i\rangle\langle b, e_i\rangle\langle v, a\rangle.\end{aligned}$$

Summing over i yields

$$\sum_{i=1}^{n+1}\langle A_i \wedge \nabla_v A_i, a \wedge b\rangle = \langle a, x\rangle\langle v, b\rangle - \langle x, b\rangle\langle v, a\rangle = 0. \tag{5.5}$$

Let B be a vector field on $\mathbb{S}^n$; by (5.1), $\nabla_B A_i = -\langle x, e_i\rangle B$. Using $\mathcal{L}_{A_i}B = \nabla_{A_i}B - \nabla_B A_i$ and combing with (5.2) and (5.3), we get that

$$\sum_{i=1}^{m} \mathrm{div}(A_i)\mathcal{L}_{A_i}B = -nB. \tag{5.6}$$

Finally we notice that by (5.4)–(5.6), the vector fields $A_1, \cdots, A_{n+1}$ satisfy the conditions (a)–(c) but not (d) in Section 3.

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